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Chaos control in the uncertain Duffing oscillator

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Abstract

A robust control scheme is developed for the well-known chaotic Duffing equation subject to uncertainties. Based on Lyapunov stability theory, sufficient conditions for tracking a smooth periodic orbit have been analyzed theoretically and numerically. The robust feedback control law is composed of a dynamic compensator and a linearizing control law including estimation of uncertainties. The control time is explicitly computed. Simulation results are presented to verify the validity of the proposed scheme.

1. Introduction

The control of chaotic systems has received increasing attention in recent years [1–7]. Ott et al. [1] provided the first strategy to stabilize periodic orbits embedded in a chaotic attractor. Since then, several techniques have been applied for the control of a variety of chaotic systems. The occurrence of a particular type of control may depend on the structure of the underlying dynamical system considered. One approach is based on backstepping design techniques [2,3]. Another exploits Lyapunov stability theory, enabling nonadaptive and adaptive designs to be proposed for several type [4–7].

The purpose of this paper is to make a novel contribution to the Lyapunov control of chaotic continuous-time systems. Because of the tremendous complexity of chaotic dynamics, the problem is restricted to the Duffing equation which has been investigated as a benchmark chaotic system for vibration in several papers [8–14]. The Duffing equation is of great interest because it has many applications in physics and engineering. Indeed, systems as diverse as the forced pendulum, electrical circuits, oscillations in plasma, magneto-elastic mechanical systems, suppressors of vibration, vibration absorbers, etc., are all governed by the Duffing equation. The literature studying the Duffing equation is extensive, and it is well known that the solutions of Duffing's equation are already complicated enough to exhibit multiple periodic solutions, quasiperiodic orbits, and chaos. A good guide to the properties of the Duffing equation is the book by Nayfeh and Mook [13].

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Highly recommended as a pioneering work in nonlinear dynamics is the book by Hayashi, which deals almost exclusively with the Duffing equation [14].

In this paper, the general form of Duffing's equation with an additive control input u is considered:

$$\ddot{x} + p\dot{x} + p_1 x + p_2 x^3 = u + q\cos\omega t, \tag{1}$$

where p, p_1, p_2, q and ω are nonzero constant parameters. The control u is added so as to order or guide the chaotic dynamics to meet our specific requirements. It is assumed that uncertainties such as modelling errors, parameter variations, perturbing external forces and time lags are present in the model. That is, the problem of robust control of chaotic signals against model uncertainties will be addressed. In this work, the authors are interested in driving the state x to an appropriately defined reference signal x_d at a finite time. This issue is widely known as the *tracking problem* in the scientific community.

Previous works [15–24] have presented interesting results on the control of the forced Duffing equation (1). For instance, Lyapunov methods [16], adaptive strategies [18–21], time-delay coordinates schemes [21], and robust asymptotic linearization [23,24] have been reported. The seemingly first solution developed by Chen and Dong [15] is applicable to Duffing's equation only when $p>0$ and $p_2 = 1$. In addition, their result solves the local control problem, i.e., only those trajectories of (1) starting from a small neighborhood of the desired reference orbit can be asymptotically controlled so as to stay on the desired trajectory. The assumption that $p>0$ was relaxed by Nijmeijer and Berghuis [16] following the classical Lyapunov direct method. Global control results were obtained in Refs. [16,17]. However, in order to design a control law using Lyapunov methods, one requires a priori knowledge about the model, which can be a restrictive condition (for example, in the case of chaos synchronization applied to secure communication, the transmitter model is not exactly known [22]). Adaptive control schemes can be considered as a form where a reference model tracks the dynamics of the system. These techniques perform well and allow the control of Duffing's equation, even though the parameters are not all known [21] or are time-varying [20]. Nevertheless, such strategies have drawbacks: the parameters of the model must be known. This requirement results in very complex feedback schemes. The controller based on time-delay coordinates [21] is robust against modelling errors and perturbing external forces; however, since the control scheme uses a discrete velocity estimator, it is very sensitive to noisy measurements. As in adaptive schemes, robust asymptotic linearization is composed of two parts: an uncertainty estimator and a linearizing control law [24]. The robust asymptotic feedback has the following advantages in relation to adaptive control schemes: (i) the order of the proposed controller does not increase with the number of parameters and (ii) if the system is nonlinear in its parameter structure, the robust asymptotic feedback does not change because the controller does not require information about system's parameters. The main difference between adaptive schemes and robust asymptotic linearization is that the latter does not require a priori knowledge about the model parameterization; however, it is not robust against noisy measurements.

The contributions of this paper are twofold. First, a new solution for the control problem will be proposed using advanced nonlinear control theory. The proposed strategy is an input–output control scheme which comprises an uncertainty estimator and an asymptotic linearizing-like feedback. The robust asymptotic controller is designed by means of the following procedure: (a) the uncertainties are lumped in a nonlinear function, (b) the lumped nonlinear function is interpreted as an augmented state in such a way that the extended system is dynamically equivalent to the original system, (c) in order to obtain an estimate of the augmented state, a state estimator is designed for the extended system and (d) the estimated value of the uncertainties is provided for the control law (via the estimated value of the augmented state). The above procedure yields a feedback controller which leads to chaos control in spite of modelling errors, parametric variations and/or external perturbations. It is pointed out that the control problem considered here differs from that of Refs. [18–20]. The robustness of the feedback control law against model uncertainties and time lags in the actuator is shown through numerical simulations.

The second contribution of this paper is to argue the importance of the control time in the context of chaos control. It is well known in the nonlinear community that optimization is a key word for widespread applications, and efforts should be made to fulfill optimization criteria such as the minimization of both control time and required energy input for the process. Using Lyapunov stability theory, we derive an explicit expression for the control time and show how the feedback parameters affect the convergence rate. This can be easily done by tuning the two parameters of the feedback control law.

It is hoped that the methodology developed for this peculiar chaotic system will be applicable to other types of chaotic systems such as Chua's circuits, Rössler and Lorenz systems and many other types of vibrational systems. We think that this technique provides a strong tool for control theory and is full of promise because it could be applied in a great range of problems: stabilization, observers, synchronization, etc.

2. Statement of the problem

It is a well known fact that the Duffing oscillator presents very complicated dynamics such as the coexistence of chaotic attractors with periodic orbits [15]. In practical applications, it is desirable to induce regular dynamics in the Duffing oscillator, e.g., to avoid fracture and degradation of the parts of a mechanism. Thus, the control objective is to solve the following global tracking problem: for any bounded reference trajectory whose derivatives are bounded and piecewise continuous on $[0, \infty)$, to design a feedback control law u that forces the output $y = x$ to track $y_d = x_d$ asymptotically for all $t \geqslant T \geqslant 0$ and initial conditions $(x(0), \dot{x}(0))$ despite modelling errors, parameter variations, perturbing external forces and time lags in the actuator, that is,

$$\lim_{t \to T} y(t) = y_d(t). \tag{2}$$

Using $x = x_1$ and $\dot{x} = x_2$, Eq. (1) may be rewritten as follows:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \Theta(x_1, x_2, t) + u, \end{aligned} \tag{3}$$

where $\Theta(x_1, x_2, t) = -px_2 - p_1x_1 - p_2x_1^3 + q\cos\omega t$ is a smooth nonlinear function.

In order to design a control law satisfying the control objective stated above, let us assume the following.

Assumption 1. Only the output $y = x_1$ is available for feedback.

Assumption 2. The function $\Theta(x_1, x_2, t)$ is unknown.

Some comments regarding the above assumptions are in order. Assumption 1 is realistic because in most cases only the position coordinate is available for feedback. Although its time derivative can be obtained by means of encoders, the procedure is very sensitive to noisy measurements. Assumption 2 refers to a general and practical situation because the term $\Theta(x_1, x_2, t)$ involves the uncertainties in the system. Hence, the nonlinear function $\Theta(x_1, x_2, t)$ is unknown and it is clear that it cannot be directly used in a linearizing-type of feedback. These kinds of uncertainties have previously been studied in the context of chaos control and synchronization in Refs. [20–22].

The idea in dealing with the uncertain term $\Theta(x_1, x_2, t)$ in Eq. (3) is to lump it into a new state η. Thus, let $\eta(t) = \Theta(x_1, x_2, t)$. In this way, system (3) can be rewritten as the following extended dynamically equivalent system [7]:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \eta + u, \\ \dot{\eta} &= \Xi(x_1, x_2, \eta, u, t), \end{aligned} \tag{4}$$

where

$$\Xi(x_1, x_2, \eta, u, t) = x_2\partial_1\Theta(x_1, x_2, t) + (\eta + u)\partial_2\Theta(x_1, x_2, t)$$

with $\partial_k\Theta(x_1, x_2, t) = \partial\Theta(x_1, x_2, t)/\partial x_k$, $k = 1, 2$.

For the sake of compactness, we introduce the following alternative description for system (4):

$$\begin{aligned} \dot{z} &= Az + B(\eta + u), \\ \dot{\eta} &= \Xi(z, \eta, u, t), \end{aligned} \tag{5}$$

where $z = (x_1, x_2)^T$, A and B are in the Brunovsky canonical form, i.e.,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is straightforward to prove that $\Psi(x_1, x_2, \eta, t) = \eta - \Theta(x_1, x_2, t)$ is a first integral of system (4). In order to prove this property, it suffices to show that along the trajectories of system (4), one has $\mathrm{d}\Psi(x_1, x_2, \eta, t)/\mathrm{d}t = 0$ for all $t \geqslant 0$ or equivalently $x_2 \partial_1 \Psi(x_1, x_2, \eta, t) + (\eta + u)\partial_2 \Psi(x_1, x_2, \eta, t) + \dot{\eta}\partial_\eta \Psi(x_1, x_2, \eta, t) + \partial_t \Psi(x_1, x_2, \eta, t) = 0$ (with $\partial_k \Psi(x_1, x_2, \eta, t) = \partial\Psi(x_1, x_2, \eta, t)/\partial x_k$, $k = 1, 2$, $\partial_\eta \Psi(x_1, x_2, \eta, t) = \partial\Psi(x_1, x_2, \eta, t)/\partial\eta$ and $\partial_t \Psi(x_1, x_2, \eta, t) = \partial\Psi(x_1, x_2, \eta, t)/\partial t$). This is automatically satisfied because $\partial_\eta \Psi(x_1, x_2, \eta, t) = 1$ and $\dot{\eta} = -x_2 \partial_1 \Psi(x_1, x_2, \eta, t) - (\eta + u)\partial_2 \Psi(x_1, x_2, \eta, t) - \partial_t \Psi(x_1, x_2, \eta, t)$. Hence, system (4) is dynamically equivalent to system (3). This implies that the augmented state η provides the dynamics of the uncertain function $\Theta(x_1, x_2, t)$.

3. A feedback control law with estimation of uncertainties

Let $e \in \mathbb{R}^2$ be the tracking error vector whose components are given by $e_1 = x_1 - x_{1d}$ and $e_2 = x_2 - x_{2d}$ and $N(\theta) = N^T(\theta)$ a positive definite matrix, solution of the differential matrix equation:

$$\frac{\mathrm{d}X}{\mathrm{d}\theta} = \frac{1}{\beta}\,\theta^{(1/\alpha)-1}[-AX - XA^T - \beta\theta^{-1/\alpha}X + BB^T]. \tag{6}$$

Here α and β are positive constants and θ is the unique positive solution of the equation:

$$\theta^{1+3/\alpha} = \sum_{i,j=1}^{2} [F(\alpha,\beta)]_{ij}^{-1}\,\theta^{(1/2)(i+j-2)} e_i e_j, \tag{7}$$

where $[F(\alpha,\beta)]_{ij}^{-1}$ are the elements of the inverse matrix of $[F(\alpha,\beta)]$ which elements are given by

$$[F(\alpha,\beta)]_{ij} = \frac{(-1)^{i+j}(\alpha/\beta)^{5-i-j}(4-i-j)!}{(2-i)!(2-j)![(\alpha+1)\cdots(\alpha+5-i-j)]}, \quad i,j = 1,2. \tag{8}$$

Moreover, we have

$$[N(\theta)]_{ij} = \frac{(-1)^{i+j}((\alpha/\beta)\theta^{1/\alpha})^{5-i-j}(4-i-j)!}{(2-i)!(2-j)![(\alpha+1)\cdots(\alpha+5-i-j)]}, \quad i,j = 1,2. \tag{9}$$

Eq. (6) will be useful in proving that the dynamics of the closed-loop system is globally asymptotically stable. This is motivated by the fact that the proposed control scheme is based on the use of bounded positive functions that are nonincreasing along the solutions of the closed-loop system [25,26].

Now, let us consider the following linearizing-like control law:

$$u = \dot{x}_{2d} - \eta - \tfrac{1}{2}B^T N^{-1}(\theta)e, \tag{10}$$

where $N^{-1}(\theta)$ is the inverse matrix of $N(\theta)$ and $e = (e_1, e_2)^T$.

Substitution of the linearizing-like controller (10) into Eq. (5) leads to

$$\begin{aligned} \dot{e} &= (A - \tfrac{1}{2}BB^T N^{-1}(\theta))e, \\ \dot{v} &= \Gamma(e, \eta, u, t), \end{aligned} \tag{11}$$

where $v = \eta - \eta_d$ and $\Gamma(e, \eta, u, t) = \Xi(x_1, x_2, \eta, u, t) - \Xi(x_{1d}, x_{2d}, \eta, u, t)$.

We are now ready to state the first main result of this paper.

Theorem 1. *Consider the tracking error system* (11). *If* $\alpha > 1$ *and* $\beta > 0$, *the tracking error* $e(t)$ *converges asymptotically to zero at a finite time*

$$T = \frac{\alpha}{\beta}\,\theta^{1/\alpha}(e_0), \tag{12}$$

where $e_0 = e(0)$ *is the initial condition of* $e(t)$.

Proof. Without loss of generality, assume that $e(t)$ belongs to the interval $[0, T]$ so that $\theta \neq 0$ and the matrices $N(\theta)$ and $N^{-1}(\theta)$ exist. Consider the following Lyapunov function candidate:

$$\theta(e) = e^{\mathrm{T}} N^{-1}(\theta) e. \tag{13}$$

Its time derivative along the trajectories of system (11) satisfies

$$\dot{\theta}(e) = \left\langle \frac{\partial \theta}{\partial e}, \dot{e} \right\rangle, \tag{14}$$

where $\langle .,. \rangle$ is the inner product of two vectors. From the equation $G(\theta, e) = \theta(e) - \langle N^{-1}(\theta)e, e\rangle$, one has that $\mathrm{d}G = G'_\theta(\partial\theta/\partial e) + G'_e = 0$, where

$$G'_e = \frac{\partial G}{\partial e}(\theta, e) = N^{-1}(\theta)e$$

and

$$G'_\theta = \frac{\partial G}{\partial \theta}(\theta, e) = e^{\mathrm{T}}\left[\frac{1}{\theta} N^{-1}(\theta) - \frac{\mathrm{d}}{\mathrm{d}\theta} N^{-1}(\theta)\right] e.$$

This implies that $\partial\theta/\partial e = -G'_e/G'_\theta$. With this in mind, $\dot{\theta}(e)$ will become

$$\begin{aligned} \dot{\theta}(e) &= -\left\langle \frac{G'_e}{G'_\theta}, \left(A - \frac{1}{2} BB^{\mathrm{T}} N^{-1}(\theta)\right) e \right\rangle, \\ &= \frac{1}{G'_\theta} e^{\mathrm{T}}[A^{\mathrm{T}} N^{-1}(\theta) + N^{-1}(\theta)A + N^{-1}(\theta)BB^{\mathrm{T}} N^{-1}(\theta)]e. \end{aligned} \tag{15}$$

Now, using Eq. (6), one may easily prove that

$$e^{\mathrm{T}}[A^{\mathrm{T}} N^{-1}(\theta) + N^{-1}(\theta)A + N^{-1}(\theta)BB^{\mathrm{T}} N^{-1}(\theta)]e = \beta\theta^{1-1/\alpha} G'_\theta. \tag{16}$$

Then, we get

$$\dot{\theta}(e) = -\beta\theta^{1-1/\alpha}(e) \tag{17}$$

which is negative definite if $\alpha > 1$ and $\beta > 0$ [27]. This means that if $\alpha > 1$ and $\beta > 0$, the tracking error $e(t)$ converges asymptotically to zero. Convergence of $v(t)$ to zero follows from the fact that the closed-loop system is in a cascade form [7]. From Eq. (3), it is known that $\Theta(x_1, x_2, t)$ is smooth. Then, the control dynamics is given by

$$\dot{u} = \ddot{x}_{2d} - \Xi(x_1, x_2, \eta, u, t) - \tfrac{1}{2} B^{\mathrm{T}} \dot{N}^{-1}(\theta)e - \tfrac{1}{2} B^{\mathrm{T}} N^{-1}(\theta)\dot{e}.$$

Since $\Xi(x_1, x_2, \eta, u, t)$ is a smooth function, $\dot{u}$ is also a smooth function. Consequently, $\Psi(x_1, x_2, \eta, u, t) = \eta - \Theta(x_1, x_2, t)$ is a first-integral of the closed-loop system. Then, from Eq. (10), the augmented state becomes $\eta = \dot{x}_{2d} - u - \frac{1}{2} B^{\mathrm{T}} N^{-1}(\theta)e$. Hence, the augmented state η is bounded and its dynamics is also bounded. In addition, since $v = \eta - \eta_d$, v is also bounded. Finally, since $e(t)$ asymptotically converges to zero, $v(t)$ also asymptotically converges to zero.

To compute the control time, we have to follow the time trajectory of the closed-loop system (11). In this case, the control objective is achieved when the tracking error $e(t)$ is zero for all $t \geqslant T_{\mathrm{con}} \geqslant 0$. Let us integrate Eq. (17) to get $\theta^{1/\alpha} = (1/\alpha)(-\beta t + c)$, where c is an integration constant. Note that if $e_0 \neq 0$ then $\theta(e_0) \neq 0$, whence $c = \alpha\theta^{1/\alpha}(e_0)$. With this in mind, since $\theta(e) = 0$ at $t = T$, one may easily prove that the expression for the control time is given by Eq. (12). This implies that $e(T) = 0$. On the other hand, according to LaSalle invariance principle [28], the largest invariant set contained in $E = \{e \in \mathbb{R}^2, \dot{\theta}(e) = 0\}$ is the manifold $e = 0$. Thus, since $e(T) = 0$, one can conclude that the tracking error $e(t)$ remains at zero for all $t \geqslant T \geqslant 0$ since the manifold $e = 0$ is the largest invariant set of $\mathbb{R}^2$. This achieves the proof. □

If the conditions $\alpha > 1$ and $\beta > 0$ are not satisfied, $\dot{\theta}(e)$ is not negative definite and the control process is unstable. Here, the instability means that $e(t)$ never goes to zero, but has a bounded oscillatory behavior or goes to infinity. In addition, if $e_0 = 0$, θ will become 0 too so that $T = 0$. In this case, the control process is loss. Thus, the conditions $e_0 \neq 0$, $\alpha > 1$ and $\beta > 0$ are required to avoid loss of instability during the control process.

The following points should be noted. (i) Given the feedback parameters α and β, it is not immediately apparent how one chooses the function θ so that the control objective (2) is satisfied. Furthermore, it is not easy to find the analytic solutions of Eq. (7). Fortunately, this equation can be solved numerically. (ii) The α, β-parameterization of the feedback control law (10) provides a simple tuning procedure. From Eq. (12), one can observe that for β fixed, if α increases, then $\theta^{1/\alpha}(e_0)$ decreases so that $\alpha\theta^{1/\alpha}(e_0)$ increases. This means that the control time T increases with α. Also, according to Eq. (7), it is found that the control time can be expressed as the inverse of the degrees of β. Therefore, for α fixed, if β increases, then the control time T decreases. Hence, the analysis points to how the control time can be minimized. This is of great practical interest, since the control can be affected as fast as desired, just depending on the feedback parameters α and β.

Nevertheless, the linearizing-like feedback (10) is not physically realizable because it requires measurements of the velocity x_2 and a perfect knowledge of the nonlinear term $\Theta(x_1, x_2, t)$. Because of Assumptions 1 and 2, the linearizing-like feedback (10) must be modified in such a way as to encompass consideration of modelling errors and parameter perturbations. We therefore use the estimation of $\Theta(x_1, x_2, t)$ in such a way that the main characteristics of the linearizing-like feedback (10) are retained. An important advantage of system (4) is that the dynamics of the state η can be reconstructed from the output $y = x_1$ by the following uncertainty estimator:

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 - Lk_1\phi_1(\hat{x}_1 - x_1),\\
\dot{\hat{x}}_2 &= \hat{\eta} + u - L^2k_2\phi_2(\hat{x}_1 - x_1),\\
\dot{\hat{\eta}} &= -L^3k_3\phi_3(\hat{x}_1 - x_1),
\end{aligned} \tag{18}
$$

where L is the so-called high-gain parameter which can be interpreted as the estimation rate of uncertainties and often be chosen as a constant, k_i with $i = 1, 2, 3$ to be determined and

$$
\phi_i(\hat{x}_1 - x_1) = [\mathrm{abs}(\hat{x}_1 - x_1)]^r \,\mathrm{sign}(\hat{x}_1 - x_1), \quad r > 0, \ i = 1, 2, 3. \tag{19}
$$

Now, the second result of this paper is stated by the following theorem.

Proposition 1. *Let $\tilde{e} \in \mathbb{R}^3$ be an estimation error vector whose components are defined as follows: $\tilde{e}_1 = \hat{x}_1 - x_1$, $\tilde{e}_2 = \hat{x}_2 - x_2$ and $\tilde{e}_3 = \hat{\eta} - \eta$. For a sufficiently large value of L, the dynamics of the estimation error $\tilde{e}(t)$ decays exponentially to zero.*

Proof. Combining systems (4) and (18), the dynamics of the estimation error can be written as follows:

$$
\begin{aligned}
\dot{\tilde{e}}_1 &= \tilde{e}_2 - Lk_1\phi_1(\tilde{e}_1),\\
\dot{\tilde{e}}_2 &= \tilde{e}_3 - L^2k_2\phi_2(\tilde{e}_1),\\
\dot{\tilde{e}}_3 &= -\Xi(x_1, x_2, \eta, u, t) - L^3k_3\phi_3(\tilde{e}_1).
\end{aligned} \tag{20}
$$

In order to determine k_i, the following change of variables is considered [29]:

$$
\begin{aligned}
v_1 &= L^2[\mathrm{abs}(\hat{x}_1 - x_1)]^r \,\mathrm{sgn}(\hat{x}_1 - x_1),\\
v_2 &= L(\hat{x}_2 - x_2),\\
v_3 &= \hat{\eta} - \eta.
\end{aligned} \tag{21}
$$

Notice that

$$
\dot{v}_1 = L^2[\mathrm{abs}(\hat{x}_1 - x_1)\,\mathrm{sgn}(\hat{x}_1 - x_1)]\dot{\tilde{e}}_1.
$$

From the boundedness of the chaotic attractor, it is known that x_1 is bounded which implies that $\hat{x}_1$ must be bounded too. So, one can conclude that $\mathrm{abs}(\hat{x}_1 - x_1)\,\mathrm{sgn}(\hat{x}_1 - x_1)$ is bounded. Without loss of generality, assuming that its upper limit is ρ, which can be chosen as a large enough number, then

$$
\dot{v}_1 \leqslant \rho L^2 \dot{\tilde{e}}_1. \tag{22}
$$

So, the following dynamics of the estimation error system can be obtained:

$$\dot{v} = LD(\rho, k)v + \Omega(x_1, x_2, \eta, u, t), \tag{23}$$

where $v = (v_1, v_2, v_3)^{\mathrm{T}}$, $\Omega(x_1, x_2, \eta, u, t) = [0, \ldots, 0, \Xi(x_1, x_2, \eta, u, t)]^{\mathrm{T}}$ and

$$D(\rho, k) = \begin{bmatrix} -\rho k_1 & \rho & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix}.$$

After choosing ρ, according to experience, one can select the constants k_i, $i = 1, 2, 3$ in such a way that $D(\rho, k)$ has all its eigenvalues in the left-half-complex plane. Since $x_1(t)$ and $x_2(t)$ belong to a chaotic attractor, the function $\Xi(x_1, x_2, \eta, u, t)$ in Eq. (4) is a bounded function. Consequently, for a sufficiently large value of the high-gain parameter L, one can conclude that $v_i(t) \to 0$, $i = 1, 2, 3$. That is, the estimation error system is asymptotically stable at zero, which implies that $\hat{x}_1 \to x_1$, $\hat{x}_2 \to x_2$ and $\hat{\eta} \to \eta$. So, one can get information about unmeasurable states from $\hat{x}_1$, $\hat{x}_2$ and the uncertain function $\Theta(x_1, x_2, t)$ from $\hat{\eta}$. This achieves the proof. □

Thus, the output feedback control law (10) with the state estimations becomes

$$u = \dot{x}_{2d} - \hat{\eta} - \tfrac{1}{2} B^{\mathrm{T}} N^{-1}(\theta)\hat{e}, \tag{24}$$

where θ is the unique positive solution of the equation:

$$\theta^{1+3/\alpha} = \sum_{i,j=1}^{2} [F(\alpha, \beta)]_{ij}^{-1} \theta^{(1/2)(i+j-2)} \hat{e}_i \hat{e}_j. \tag{25}$$

We can state the main result of this paper.

Corollary 1. *Consider system* (4) *equipped with the observer-based output-feedback* (18), (24). *Then, the tracking error* $e(t)$ *converges asymptotically to zero at a finite time*

$$T = \frac{\alpha}{\beta} \theta^{1/\alpha}(\hat{e}_0), \tag{26}$$

where $\hat{e}_0 = \hat{e}(0)$ *is the initial state of* $\hat{e}(t)$.

Note that the linearizing-like feedback (24) only uses the estimation of the uncertain function $\Theta(x_1, x_2, t)$ (by means of $\hat{\eta}$) and $\hat{x}$ which are provided by the state estimator (18). The dynamical compensator (18) only uses the measurable signal $y = x_1$. So the feedback controller (24) is more physically realizable than Eq. (10).

Note also that the output feedback controller (18), (24) yields practical stabilization, i.e., system (4) converges to a ball $\mathfrak{B}$ whose radius is of order of L^{-1}. That is, $e(t) \to \mathfrak{B}(r(L^{-1}))$ as $t \to T$. In fact, the integration of the estimation error system (23) yields

$$v(t) = \exp(D(\rho, k)t)v(0) + \exp(D(\rho, k)t) \int \exp(D(\rho, k)s)\Omega(x_1, x_2, \eta, u, t)\, \mathrm{d}s.$$

Now, using the triangle and Schwartz inequalities, one has

$$\|v(t)\| \leqslant \|v(0)\exp(LD(\rho, k)t)\| + \exp(LD(\rho, k)t) \int \|\exp(-LD(\rho, k)s)\Omega(x_1, x_2, \eta, u, t)\, \mathrm{d}s\|.$$

Since $\Omega(x_1, x_2, \eta, u, t)$ is bounded,

$$\|v(t)\| \leqslant \|v(0)\exp(LD(\rho, k)t)\| + \sigma_1 \exp(D(\rho, k)t)\|v(0)\| + \sigma_2 \tag{27}$$

for any constants σ_1 and σ_2 such that $\|\Omega(x_1, x_2, \eta, u, t)\| \leqslant \sigma_1 \exp(D(\rho, k)t)\|v(0)\| + \sigma_2$. In other words, $v(t) \to \mathfrak{B}(r(L^{-1}))$, where $\mathfrak{B}(r(L^{-1}))$ is a ball with radius of the order of L^{-1}. In fact as the estimation parameter L increases, $\|v(t)\|$ decreases. Then, according to Eq. (21), one can conclude that $x_1 \to \hat{x}_1$, $x_2 \to \hat{x}_2$ and $\eta \to \hat{\eta}$. As a consequence, the feedback (24) tends to the linearizing-like controller (10). Control actions therefore counteract the nonlinear uncertainties and induce a linear behavior.

Feedback control law based on high-gain observers can induce undesirable dynamical effects such as the so-called peaking phenomenon [30]. This leads to closed-loop instabilities which are represented by time-finite escapes and large overshoots. To diminish the effect of these instabilities, the robust control law can be modified by means [31]:

$$u = \mathrm{Sat}\left\{\dot{x}_{2d} - \hat{\eta} - \tfrac{1}{2}B^{\mathrm{T}}N^{-1}(\theta)\hat{e}\right\}, \tag{28}$$

where

$$\mathrm{Sat}\{.\} = \begin{cases} = u_{\max}, & \text{if } u > u_{\max}, \\ = \dot{x}_{2d} - \hat{\eta} - \tfrac{1}{2}B^{\mathrm{T}}N^{-1}(\theta)\hat{e}, & \text{if } -u_{\max} \leqslant u \leqslant u_{\max}, \\ = -u_{\max}, & \text{if } u < -u_{\max}. \end{cases}$$

4. Simulation results

In this section, numerical simulations are given to verify the proposed robust control scheme. The parameters p, p_1, p_2, ω and q are chosen as $p = 0.4$, $p_1 = -1.1$, $p_2 = 1$, $\omega = 1.8$ and $q = 1$, respectively, in all simulations to ensure the existence of chaos in the absence of control as shown in Fig. 1. Initial conditions were arbitrarily located at the point $(x_1(0), x_2(0)) = (0.2, 0)$. Furthermore, there exists a bounded region $\Gamma \in \mathbb{R}^2$ containing the whole attractor such that no orbit of system (1) ever leaves it [15].

The control objective is to drive the output of the uncertain chaotic system (1) to the trajectory $y_d = \sin 2t$. Obviously, this desired trajectory does not belong to the embedded orbits of the strange attractor.

For the sake of clarity, note that the feedback control law (24) can be rewritten as

$$u = \dot{x}_{2d} - \hat{\eta} - \frac{(\alpha+2)(\alpha+1)(\hat{x}_1 - x_{1d})}{2((\alpha/\beta)\theta^{1/\alpha})^2} - \frac{(\alpha+2)(\hat{x}_2 - x_{2d})}{((\alpha/\beta)\theta^{1/\alpha})}, \tag{29}$$

where θ is the unique positive solution of the equation:

$$\begin{aligned}\theta^{(\alpha+3)/\alpha} = {} & \frac{2\beta}{\alpha}(\alpha+2)(\hat{x}_2 - x_{2d})^2\theta^{2/\alpha} + \frac{2\beta^2}{\alpha^2}(\alpha+2)(\alpha+1)(\hat{x}_1 - x_{1d})(\hat{x}_2 - x_{2d})\theta^{1/\alpha} \\ & + \frac{\beta^3}{\alpha^3}(\alpha+2)^2(\alpha+3)\cdot(\hat{x}_1 - x_{1d})^2,\end{aligned} \tag{30}$$

in which $\hat{x}_1$, $\hat{x}_2$ and $\hat{\eta}$ are given by the state estimator (18).

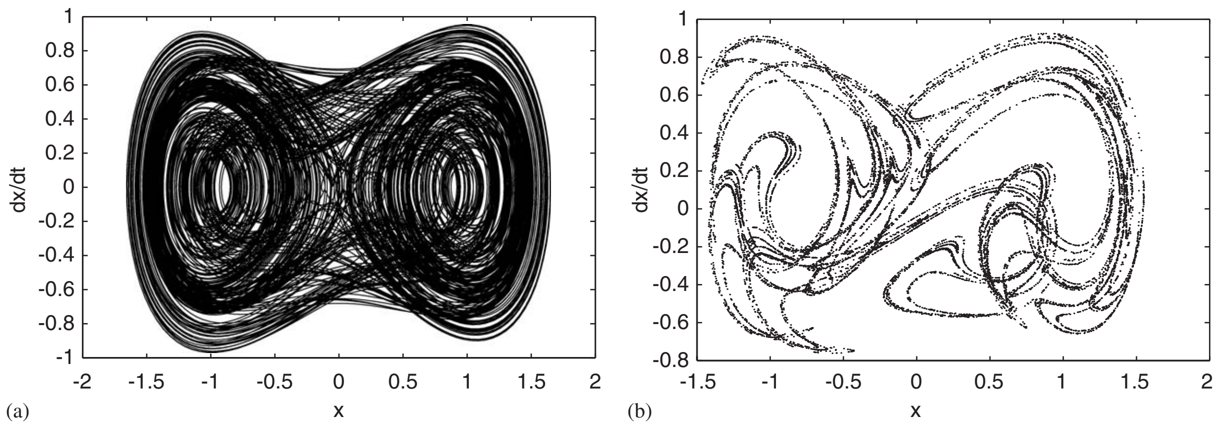

Fig. 1. Phase plane and Poincaré map of the chaotic Duffing oscillator for $u = 0$ (uncontrolled evolution).

The initial conditions of observer (18) are selected to be $(\hat{x}_1(0), \hat{x}_2(0), \hat{\eta}(0)) = (0, 0.1, 0.96)$. In this case, $e_1(0) = 0.2$ and $e_2(0) = -2$ and we are in the ideal case where our control scheme works. In this case

$$\theta(\hat{e}_0) = \left[7.22 \frac{\beta}{\alpha}(\alpha + 2)\right]^{\alpha/(\alpha+1)}$$

and

$$T = \frac{\alpha}{\beta^{\alpha/(\alpha+1)}} \left[7.22 \frac{(\alpha + 2)}{\alpha}\right]^{1/(\alpha+1)}.$$

Here one can choose $\rho = 4$. According to the parameter-choosing scheme mentioned above, one can select parameters as follows: the high-gain parameter is $L = 20$, the parameters k_i, $i = 1, 2, 3$ are $[k_1, k_2, k_3] = [1, 2, 3]$ so that the eigenvalues of matrix $D(\rho, k)$ are -2.6850, $-0.6575 \pm 2.0092\mathrm{i}$ and $r = 0.5$. The corresponding simulation results are shown in Figs. 2–5.

Figs. 2(a) and (b) show the control time T as a function of β when $\alpha = 2$, and as a function of α when $\beta = 1$. From Fig. 2(a), one can see that the control time increases with α while from Fig. 2(b), the control time decreases when β increases.

Fig. 3 shows the simulation results obtained by applying the output feedback controller (24) to the uncertain Duffing equation (1) for tracking the desired signal y_d for $\alpha = 2$ and $\beta = 1$. Fig. 3(a) presents the x component (—) together with its desired value (- - -) while Fig. 3(b) shows the $\dot{x}$ component (—) together with its desired value (- - -). Note that fairly good tracking performance is obtained. Fig. 3(c) shows the phase portrait of the closed-loop system. The control was turned on at $t = 2$ s. It is clearly evident that the attractor changed its dynamical structure in such a way that the canonical plane (x_1, x_2) has acquired a periodic structure. Such behavior is attained thanks to the fact that the compensator state $\hat{\eta}$ provides an estimate of the uncertain term $\Theta(x_1, x_2, t)$. Fig. 3(d) presents the estimated term $\hat{\eta}$ (- - -) and the current term $\eta = \Theta(x_1, x_2, t)$ (—). After a short transient, $\hat{\eta}$ evolves very closely with η. However, one can expect that the tracking errors $e_1(t) = x_1(t) - x_{1d}(t)$ and $e_2(t) = x_2(t) - x_{2d}(t)$ converge to zero. Figs. 4(a) and (b) show, respectively, the time evolution of the tracking errors $e_1(t)$ and $e_2(t)$. Note that the dynamics of the tracking error converges exactly to zero. This means that the tracking of the reference signal y_d is guaranteed by the designed feedback controller. One can also see that a fairly good tracking convergence is obtained in about 4.8 s which corresponds to the analytical value of the control time (see Fig. 2).

In order to add evidence of the effectiveness and efficiency of the proposed robust control scheme, extensive simulations have been performed to examine the effect of the feedback parameters α and β. Fig. 5 shows the output tracking error $e_1(t) = y(t) - y_d(t)$ for three different values of β when $\alpha = 2$, and for three different values of α when $\beta = 1$. As predicted by the analysis of Theorem 1, for α fixed, larger values of β or for β fixed, smaller values of α give faster convergence of y_d.

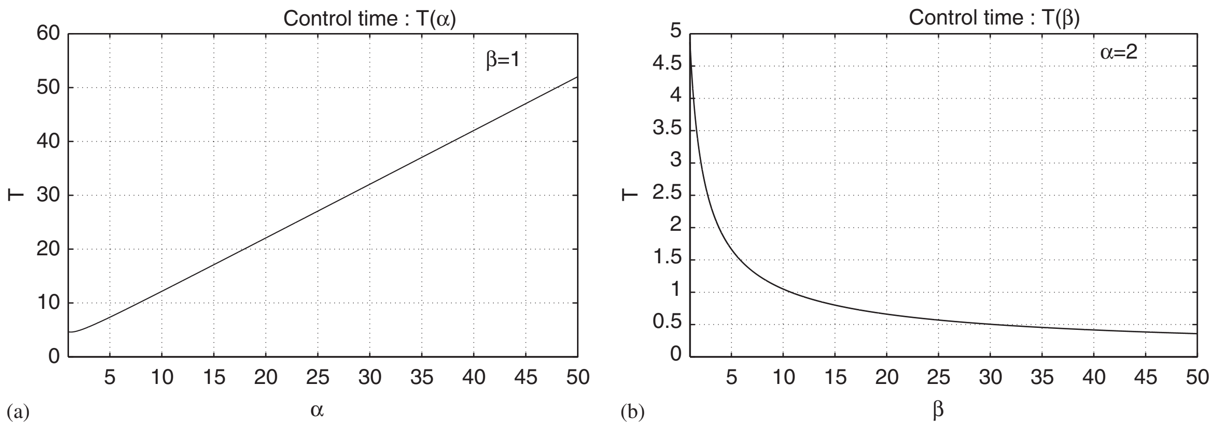

Fig. 2. Control time T: (a) as a function of α when $\beta = 1$ and (b) as a function of β when $\alpha = 2$.

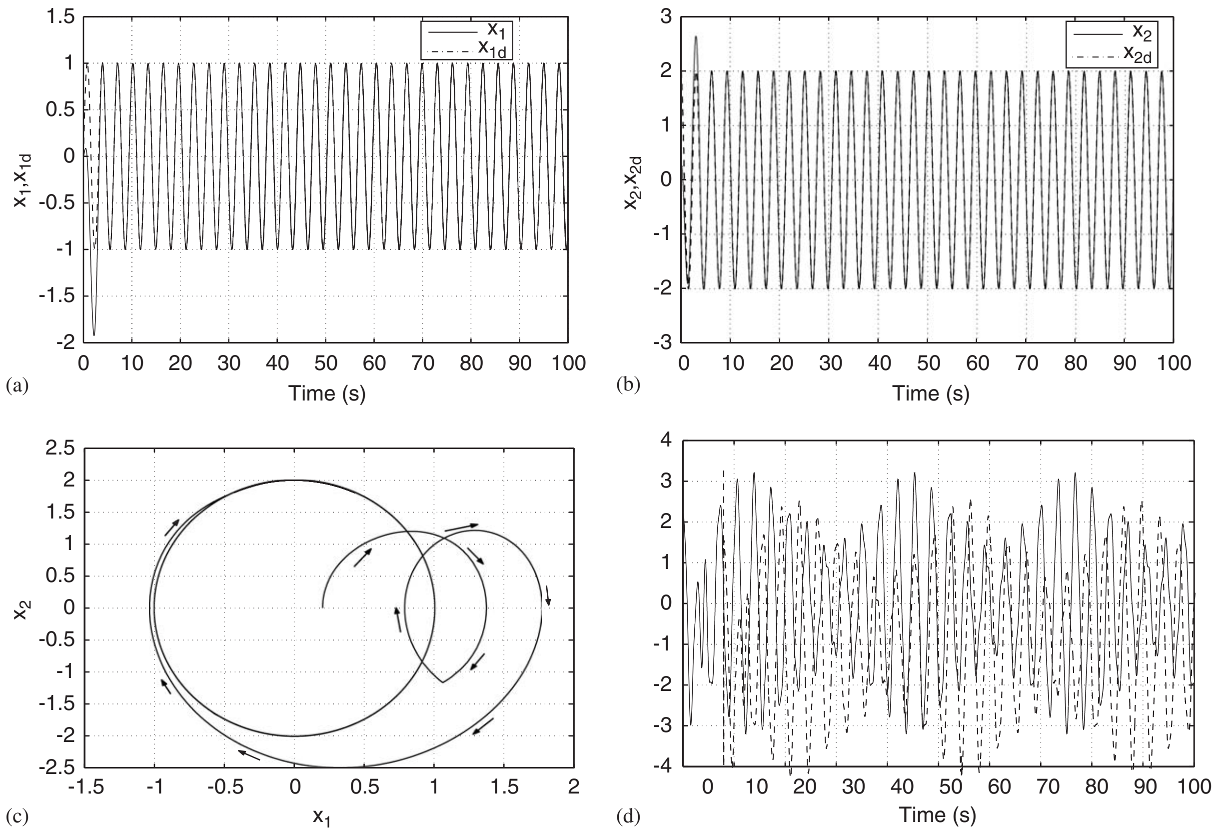

Fig. 3. State responses of the controlled Duffing system performed with $\alpha = 2$ and $\beta = 1$: (a) x component (—) together with its desired value (- - -), (b) $\dot{x}$ component (—) together with its desired value (- - -), (c) phase portrait of the closed-loop system. The control was turned on at $t = 2\,\mathrm{s}$ and (d) evolution of the current value of $\eta = \Theta(x_1, x_2, t)$ (—) and its estimated value $\hat{\eta}$ (- - -).

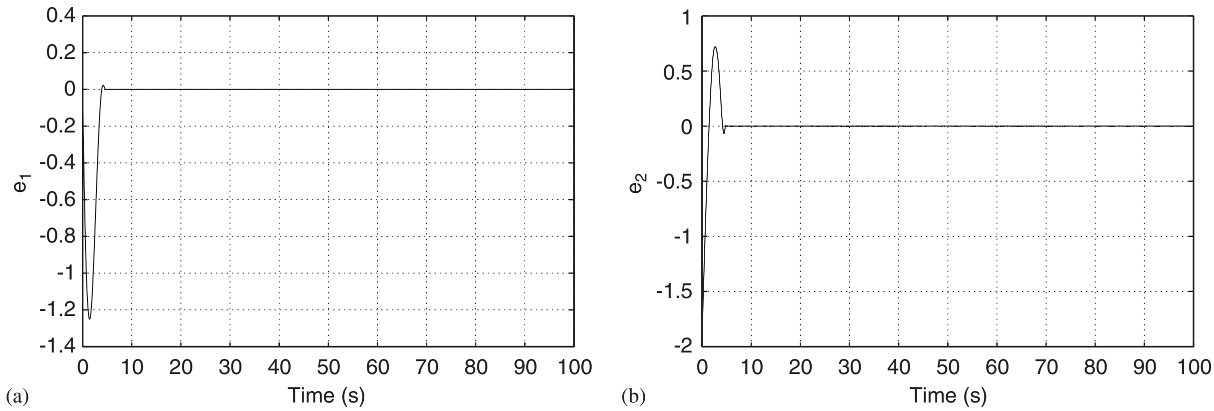

Fig. 4. Time evolution of the tracking errors: (a) $e_1(t) = x_1(t) - x_{1d}(t)$ and (b) $e_2(t) = x_2(t) - x_{2d}(t)$.

5. Conclusion

In this paper, a robust control scheme was presented for the well-known chaotic Duffing equation subject to uncertainties. By assuming that the exact model of the system is not known and that the position coordinate is the only state variable available for measurements, a feedback control law which comprises a linearizing-like

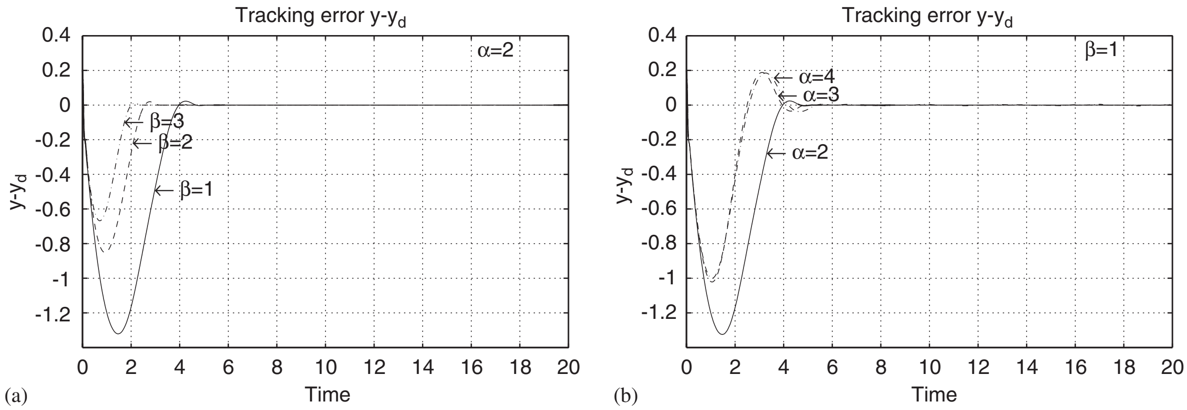

Fig. 5. Output tracking error $e_1(t) = y(t) - y_d(t)$: (a) for three different values of β when $\alpha = 2$ and (b) for three different values of α when $\beta = 1$.

feedback and an uncertainty estimator was designed so that the output signal can asymptotically track any smooth reference trajectory. An explicit expression for the control time was given in terms of two parameters for which an arbitrary convergence rate of the tracking error can be prescribed. Numerical results illustrate the efficiency of the proposed control methodology.

Although in this paper the implementation is performed via numerical simulations, it is not hard to see that the physical application of the proposed adaptive feedback can be performed. In fact, experimental studies are in progress and the results will be reported elsewhere. It is our belief that the proposed control strategy can be extended to systems of higher dimension. In principle, the proposed feedback structure can be carried out to effect the synchronization of chaotic systems. This procedure has already been used to perform the synchronization of strictly different chaotic oscillators [32]. In the present paper, the procedure presented in Ref. [32] has been generalized in the context of chaos suppression.

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